The inverse conductivity problem with an imperfectly known boundary in three dimensions

Ville Kolehmainen[*], Matti Lassas[†], Petri Ola[‡]

Abstract. We consider the inverse conductivity problem in a strictly convex domain whose boundary is not known. Usually the numerical reconstruction from the measured current and voltage data is done assuming the domain has a known fixed geometry. However, in practical applications the geometry of the domain is usually not known. This introduces an error, and effectively changes the problem into an anisotropic one. The main result of this paper is a uniqueness result characterizing the isotropic conductivities on convex domains in terms of measurements done on a different domain, which we call the model domain, up to an affine isometry. As data for the inverse problem, we assume the Robin-to-Neumann map and the contact impedance function on the boundary of the model domain to be given. Also, we present a minimization algorithm based on the use of Cotton–York tensor, that finds the pushforward of the isotropic conductivity to our model domain, and also finds the boundary of the original domain up to an affine isometry. This algorithm works also in dimensions higher than three, but then the Cotton–York tensor has to replaced with the Weyl–tensor.

1. Introduction. We consider the electrical impedance tomography problem (EIT for short), i.e. the determination of the unknown isotropic conductivity distribution inside a domain in $\mathbb{R}^3$, for example the human thorax, from voltage and current measurements made on the boundary. Mathematically this is formulated as follows: Let Ω be the measurement domain, and denote by γ the bounded and strictly positive function describing the conductivity in Ω. The voltage potential u satisfies in Ω the equation

$$(1.1) \qquad \nabla \cdot \gamma \nabla u = 0.$$

To uniquely fix the solution u it is enough to give its value on the boundary. Let this be f. In the idealized case, when the contact impedance of the measurement device is zero, one measures for all voltage distributions $u|_{\partial M} = f$ on the boundary the corresponding current

[*]Department of physics, University of Kuopio, P.O.Box 1627, 70211 Finland
[†]Department of mathematics, 02015 Helsinki University of Technology, Finland
[‡]Department of mathematics, 00014 University of Helsinki, Finland

flux through the boundary, $\gamma \partial y / \partial \nu$, where ν is the exterior unit normal to $\partial \Omega$. Mathematically this amounts to the knowledge of the Dirichlet–Neumann map Λ corresponding to γ, i.e. the map taking the Dirichlet boundary values to the corresponding Neumann boundary values of the solution to (1.1),

$$\Lambda : \quad u|_{\partial M} \mapsto \gamma \frac{\partial u}{\partial \nu}$$

The Calderón's inverse problem is then to to reconstruct γ from Λ. The problem was originally proposed by Calderón [6] in 1980 and then solved in dimensions three and higher for isotropic conductivities which are C^∞–smooth in [35] and [24]. The smoothness requirements have been since relaxed, and currently the best known result is [27] with unique determination of conductivities in $W^{3/2,\infty}$, see also [10] for a somewhat different approach to the lack of smoothness. In two dimensions the first global result is due to Nachman ([25]), and later Astala and Päivärinta showed in [4] that uniqueness holds also for general isotropic L^∞–conductivities. For the corresponding anisotropic case, see [3, 20, 21, 22] and numerical implementations of the methods with simulated and real data, see [13, 31, 32].

Assuming that the measured Dirichlet-to-Neumann map Λ_{meas} is given, an often used method to solve the EIT–problem is to minimize

$$\|\Lambda_{\mathrm{meas}} - \Lambda_\sigma\|^2 + \alpha \|\sigma\|_X^2$$

for σ defined in terms of some triangulation of Ω and $\| \cdot \|_X$ is some regularization norm; here Λ_σ is the Dirichlet–Neumann map corresponding to the conductivity σ. One then also fixes the geometry of Ω by assuming that it is for example a ball or an ellipsoid. Now, if our measurements have no error a Bayesian interpretation of this problem as a search of an MAP-estimate suggests that $\alpha = 0$. Usually, the given data Λ_{meas} does not correspond to any isotropic conductivity in the model domain. The reason for this is that there is no conformal map deforming the original domain to the model domain. Therefore, in solving the minimization problem we obtain an incorrect solution σ. This means that a systematic error in modeling causes a systematic error to the reconstruction. In particular, if we consider linearization $\gamma = \gamma_0 + \varepsilon \gamma_1$ where γ_0 is given known background conductivity and ε is small, it seems that a localized perturbation γ_1 gives a reconstruction $\sigma = \gamma_0 + \varepsilon \sigma_1$ where the reconstructed perturbation σ_1 is not localized. This is clearly seen in brain-activity measurements, see [15] and [16].

This work is continuation of [18] where the corresponding question in two dimensions was studied: we proved that on the model domain there is a unique (anisotropic) conductivity with minimal anisotropy. This follows from a result of Strebel saying that among all quasiconformal self-maps of the unit disk with a fixed boundary value there is a unique one with minimal complex dilation. In higher dimensions there are several new issues. First of all, the non-uniqueness due to anisotropy is not understood, except in the case when both the domain and the conductivity function are the real analytic ([22], [23]). Also, as we already mentioned, in the plane case one could use the theory of quasiconformal maps to break the non-uniqueness. The higher dimensional analogue of this is unknown. Finally, there is no analogue of the Riemann mapping theorem that we could use.

The structure of this paper is the following. In the first part, consisting of sections 2–4, we present the uniqueness results we have on the problem. It is worth noting that we choose

to work with the Robin–to–Neumann (RN) map instead of the Dirichlet–to–Neumann (DN) map described above. Mathematically they are equivalent, as we will show, but the RN–map is a better model for the actual measurement configuration since it takes into account the contact impedances at $\partial\Omega$ [33]. Also, we assume the the function modeling the contact impedances of the electrodes is known. There are two key ideas how we compensate for our lack of understanding of the full anisotropic problem. First thing is to note that if an isotropic conductivity is pushed forward by a diffeomorphism, the resulting conductivity is still conformally flat, and in three dimensions this is equivalent with the vanishing of the Cotton–York tensor. Secondly, we assume that our original domain is strictly convex, and then the Cohn–Vossen theorem can be used to determine the original boundary $\partial\Omega$ up to rigid motions.

In the second part we develop an algorithm for finding the shape of the domain Ω and the conductivity inside using a minimization technique. Important feature is that we do not have to construct an embedding of the boundary to the Euclidean space. We plan to report on the numerical implementation of our algorithm in a separate article.

2. Measurements. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a strictly convex domain, and denote by $\gamma = (\gamma^{ij}(x))_{i,j=1}^n$ the symmetric real valued matrix describing the conductivity in Ω. We assume that the matrix is bounded from above and from below, that is, for some $C, c > 0$ we have

$$(2.2) \qquad c\|\xi\|^2 \leq \langle \xi, \gamma(x)\xi \rangle \leq C\|\xi\|^2, \quad \text{for all } x \in \Omega.$$

We will state the precise smoothness of γ later. We start by consider the EIT problem with continuous boundary data. Instead of the Dirichlet-to-Neumann map we will use the Robin-to-Neumann map defined below that corresponds better to the measurements done in practice. We discuss later in this section the relation of the continuous model and the electrode measurements made in practice.

For the electrical potential u we write the model

$$(2.3) \qquad \nabla \cdot \gamma \nabla u = 0, \qquad x \in \Omega,$$
$$(2.4) \qquad (z\nu \cdot \gamma \nabla u + u)|_{\partial\Omega} = h,$$

where h is the Robin-boundary value of the potential and z is a function describing the contact impedance on the boundary. The contact impedance models the impedance that is caused by electro-chemical phenomena at the interface of the skin and the measurement electrodes in practical measurements [7].

In mathematical terms, the perfect boundary measurements are modeled by the Robin-to-Neumann map $R = R_{z,\gamma}$ given by

$$R : h \mapsto \nu \cdot \gamma \nabla u|_{\partial\Omega}$$

that maps the potential on the boundary to the current across the boundary. Next we relate this continuous model for measurements done in practice.

The physically realistic measurements are usually modeled by the following *complete electrode model* (see [7, 33]): Let $e_j \subset \partial\Omega$, $j = 1, \ldots, J$ be disjoint open sets of the

boundary modeling the electrodes that are used for the measurements. Let u solve the equation

$$\nabla \cdot \gamma \nabla v = 0 \qquad \text{in } \Omega, \tag{2.5}$$

$$z_j \nu \cdot \gamma \nabla v + v|_{e_j} = V_j, \tag{2.6}$$

$$\nu \cdot \gamma \nabla v|_{\partial\Omega \setminus \cup_{j=1}^J e_j} = 0, \tag{2.7}$$

where V_j are constants representing electric potentials on electrode e_j. Then, one measures the currents observed on the electrodes, given by

$$I_j = \frac{1}{|e_j|} \int_{e_j} \nu \cdot \gamma \nabla v(x) \, ds(x), \quad j = 1, \ldots, J.$$

Thus the electrode measurements are given by map $E : \mathbb{R}^J \to \mathbb{R}^J$, $E(V_1, \ldots, V_J) = (I_1, \ldots, I_J)$. We say that E is the electrode measurement matrix for $(\partial\Omega, \gamma, e_1, \ldots, e_J, z_1, \ldots, z_J)$.

The complete electrode model can alternatively be defined as follows: the Robin-to-Neumann map R_η is given by $R_\eta f = \nu \cdot \gamma \nabla u|_{\partial\Omega}$ where u is the solution of

$$\nabla \cdot \gamma \nabla u = 0 \qquad \text{in } \Omega \tag{2.8}$$

$$z\nu \cdot \gamma \nabla v + \eta v|_{\partial\Omega} = h,$$

where $z \in C^\infty(\partial\Omega)$ is such that its restriction to the electrode e_j is equal to the constant z_j and $\eta = \sum_{j=1}^J \chi_{e_j}$, where χ_{e_j} is the characteristic function of electrode e_j.

We associate to the electrode measurement matrix and to the complete electrode model also the corresponding quadratic forms $E : \mathbb{R}^J \times \mathbb{R}^J \to \mathbb{R}$ and $R_\eta : H^{-1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) \to \mathbb{R}$ given by

$$E[V, \widetilde{V}] = \sum_{j=1}^J (EV)_j \widetilde{V}_j |e_j|, \quad R_\eta[h, \widetilde{h}] = \int_{\partial\Omega} (R_\eta h) \widetilde{h} \, ds. \tag{2.9}$$

These have the following simple relation to each other: Let $S = \mathrm{span}(\chi_{e_j} : j = 1, \ldots, J) \subset H^{-1/2}(\partial\Omega)$ and define $M : V = (V_j)_{j=1}^J \mapsto \sum_{j=1}^J V_j \chi_{e_j}$ to be a map $M : \mathbb{R}^J \to S$. Then

$$E[V, \widetilde{V}] = R_\eta[MV, M\widetilde{V}]. \tag{2.10}$$

By (2.10), the electrode measurement matrix can be viewed as the discretization of the form R_η. By increasing the number of the electodes and making the gaps between them smaller, we can assume that $\eta \to 1$. In this case R_η approximates the Robin-to-Neumann map $R_{\gamma,z}$. Note that $E(V, V)$ corresponds to the power needed to maintain the voltages V in electrodes.

In practical EIT experiments, one places a set of measurement electrodes on the boundary $\partial\Omega$, e.g., around the chest of the patient. All the traditional approaches to the numerical EIT reconstruction assume that the shape of the domain Ω is known and the only unknown is the conductivity γ. However, in most EIT experiments the boundary of the body Ω is

not known accurately and since there are no practically reliable measurement methods available for the determination of the boundary, the EIT image reconstruction problem is typically solved using an approximate model domain $\widetilde{\Omega}$, which represents our best guess for the shape of the true body Ω. However, it has been noticed that the use of slightly incorrect model for the body Ω in the numerical reconstruction can lead to serious artefacts in reconstructed images [16, 1, 15]. This situation is our paradigm for the EIT problem when the boundary is unknown. Next we analyze how the deformation of the domain affects measurements.

3. Deformations of the domain.

In this section we analyze the behavior of the electrode models under a diffeomorphism. Let's consider first the Robin–to–Neumann map R. The corresponding quadratic form, which we still denote by R, is given on the diagonal by

$$(3.11) \quad R[h, h] = \int_{\partial\Omega} (u + z\nu \cdot \gamma\nabla u)\nu \cdot \gamma\nabla u \, dS_E = \int_{\Omega} \gamma\nabla u \cdot \nabla u \, dx + \int_{\partial\Omega} z \, |\nu \cdot \gamma\nabla u|^2 \, dS_E$$

where $h \in H^{-1/2}(\partial\Omega)$, u solves (2.8), and dS_E is the Euclidean volume form (or area) of $\partial\Omega$. The value $R[h, h]$ corresponds to the power needed to maintain the current h on the boundary. From the mathematical viewpoint, using the (incorrect) model domain $\widetilde{\Omega}$ instead of the original domain Ω can be viewed as a deformation of the original domain. Thus, let us next consider what happens to the conductivity equation when the domain Ω is deformed to $\widetilde{\Omega}$. Assume that $F : \Omega \to \widetilde{\Omega}$ is a sufficiently smooth orientation preserving map with sufficiently smooth inverse $F^{-1} : \widetilde{\Omega} \to \Omega$. Let $f : \partial\Omega \to \partial\widetilde{\Omega}$ be the restriction of F on the boundary. When u is a solution of $\nabla \cdot \gamma\nabla u = 0$ in Ω, then $\widetilde{u}(\widetilde{x}) = u(F^{-1}(\widetilde{x}))$ and $\widetilde{h}(x) = h(f^{-1}(x))$ satisfy the conductivity equation

$$(3.12) \qquad\qquad \nabla \cdot \widetilde{\gamma}\nabla\widetilde{u} = 0, \quad \text{in } \widetilde{\Omega},$$
$$\widetilde{z}\widetilde{\nu} \cdot \widetilde{\gamma}\nabla\widetilde{u} + \widetilde{u}|_{\partial\widetilde{\Omega}} = \widetilde{h}.$$

Here $\widetilde{h}(x) = h(f^{-1}(x))$, $\widetilde{\nu}$ is the unit normal vector of $\partial\widetilde{\Omega}$, $\widetilde{z}$ is the deformed contact impedance and $\widetilde{\gamma}$ is the conductivity

$$(3.13) \qquad\qquad \widetilde{\gamma}(x) = \left.\frac{F'(y) \, \gamma(y) \, (F'(y))^{\mathrm{T}}}{|\det F'(y)|}\right|_{y=F^{-1}(x)},$$

where $F' = DF$ is the Jacobian of the map F. Note that even if γ is isotropic, i.e., scalar valued the deformed conductivity $\widetilde{\gamma}$ can be anisotropic i.e., matrix valued.

To determine the deformed contact impedance $\widetilde{z}$, we consider the corresponding invariant $(n-1)$-form

$$J := \nu \cdot \gamma\nabla u \, dS_E \in \Omega^{n-1}(\partial\Omega)$$

corresponding to the current flux through the boundary. Next we denote $\widetilde{x} = F(x)$. A straightforward application of chain rule gives that

$$\widetilde{\nu} \cdot \widetilde{\gamma}\nabla\widetilde{u}|_{\partial\widetilde{\Omega}} = \left((\det DF)^{-1}\nu \cdot \nabla u\right) \circ f^{-1}|_{\partial\widetilde{\Omega}}$$

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since F was orientation preserving and DF is the Jacobian of F in boundary normal coordinates associated to the surface $\partial\Omega \subset \mathbb{R}^n$. In these coordinates $\det DF|_{\partial\Omega} = \det Df$, where $\det Df$ is the determinant of the differential of the the boundary map $f : \partial\Omega \to \partial\widetilde{\Omega}$. We note that $(\det Df \circ f^{-1})f_*(dS_E) = d\widetilde{S}_E$, where dS_E and $d\widetilde{S}_E$ are Euclidean volume forms of $\partial\Omega$ and $\partial\widetilde{\Omega}$, respectively. Hence, $z\nu \cdot \nabla u$ transforms as an invariantly defined function when the contact impedance is interpreted as a density, i.e.

$$(3.14) \qquad\qquad \widetilde{z}(\widetilde{x}) = (\det Df(x))z(x)$$

where $f(x) = \widetilde{x}$. Now we see that the boundary measurements are invariant: When $f : \partial\Omega \to \partial\widetilde{\Omega}$ is the restriction of $F : \Omega \to \widetilde{\Omega}$, we say that the map $\widetilde{R} = f_* R_{z,\gamma}$, defined by

$$((f_* R_{z,\gamma})h)(x) = (R_{z,\gamma}(h \circ f))(y)\big|_{y=f^{-1}(x)}, \quad h \in H^{1/2}(\partial\widetilde{\Omega})$$

has that property that $\widetilde{R} = R_{\widetilde{z},\widetilde{\gamma}}$. We call $\widetilde{R}$ the push forward of $R_{z,\gamma}$ by f.

Is is also worth noting that in formula (3.11) the integral over Ω, as well as the integral over the boundary are invariant because of the deformation rule (3.14) for the contact impedance z, that is, we have

$$R[h, h'] = \widetilde{R}[h \circ f^{-1}, h' \circ f^{-1}],$$

for $h, h' \in H^{-1/2}(\partial\Omega)$.

4. Uniqueness results Now we are ready to give the exact set–up of the problem we consider: We want to recover an image of the unknown conductivity γ in Ω from the measurements of Robin-to-Neumann map, and we assume *a priori* that γ is isotropic. We assume z, $\partial\Omega$ and R are not known. Instead, let $\widetilde{\Omega}$, called the model domain, be our best guess for the domain and let $f_m : \partial\Omega \to \partial\Omega_m$ be a diffeomorphism modeling the approximate knowledge of the boundary.

As the data for the inverse problem, we assume that we are given the boundary of the model domain $\partial\widetilde{\Omega}$, the function $z \circ f^{-1}$ corresponding to the contact impedance of electrodes, and the Robin-to-Neumann map $\widetilde{R} = (f_m)_* R$. Note the discrete analog of this data is to know the voltage-to-power form $V \mapsto E(V, V)$ and the contact impedances of the electrodes, but not the location of the electrodes or the boundary of the domain. It is reasonable to assume that the contact impedance $z \circ f_m^{-1}$ on the boundary of the model domain is known since we can observe and set up the contact impedances of the electrodes the way we want. Hence we have on the boundary of our model domain $\partial\widetilde{\Omega}$ a boundary map $\widetilde{R}$ that does not generally correspond to any isotropic conductivity. Furthermore, we saw above that there are many anisotropic conductivities for which Robin-to-Neumann map is the given map $\widetilde{R}$. Next we show that the existence of the "underlying" isotropic conductivity in Ω gives the uniqueness in $\widetilde{\Omega}$ up to a diffeomorphism and that the domain Ω and the isotropic conductivity on it can be uniquely determined.

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded, strictly convex, C^∞–domain. Assume that $\gamma \in C^\infty(\overline{\Omega})$ is an isotropic conductivity, $z \in C^\infty(\partial\Omega)$, $z > 0$ be a contact impedance,*

and $R_{\gamma,z}$ the corresponding Robin-to-Neumann map. Let $\widetilde{\Omega}$ be a model of the domain satisfying the same regularity assumptions as Ω, and $f_m : \partial\Omega \to \partial\widetilde{\Omega}$ be a C^∞–smooth orientation preserving diffeomorphism.

Assume that we are given $\partial\widetilde{\Omega}$, the values of the contact impedance $z(f_m^{-1}(\widetilde{x}))$, $\widetilde{x} \in \partial\widetilde{\Omega}$ and the map $\widetilde{R} = (f_m)_* R_{\gamma,z}$. Then we can determine Ω up to a rigid motion T and the conductivity $\gamma \circ T^{-1}$ on the reconstructed domain $T(\Omega)$.

We recall also that rigid motion is an affine isometry $T : \mathbb{R}^n \to \mathbb{R}^n$.

Proof. Assume we are given $\widetilde{R}$ and the values of the contact impedance, that is, the function $z(f_m^{-1}(\widetilde{x}))$. Let $F_m : \Omega \to \widetilde{\Omega}$ be an orientation preserving diffeomorphism satisfying $F_m|_{\partial\Omega} = f_m$. As noted before, $\widetilde{R} = R_{\widetilde{z},\widetilde{\gamma}}$ where $\widetilde{z}(x) = \det(Df_m)z(f_m^{-1}(x))$ is the contact impedance on $\partial\widetilde{\Omega}$ and $\widetilde{\gamma} = (F_m)_*\gamma$ is the push forward of γ in F_m. The Robin–to–Neumann map is a classical pseudodifferential operator of order zero, with principal symbol $1/\widetilde{z}$, and hence $\widetilde{R}$ determines $\widetilde{z}$. Since we assume also $z \circ f^{-1}$ known, we can determine the determinant $\det Df_m$; note that this gives the change of boundary area under deformation f_m. For the rest of the proof denote $\beta = \det Df_m$. Also, this implies that we can find the Dirichlet-to-Neumann map $\Lambda_{\widetilde{\gamma}} = (\widetilde{R}^{-1} - \widetilde{z}I)^{-1}$ on $\partial\widetilde{\Omega}$, that is, the map taking the Dirichlet boundary values to Neumann boundary values.

The Riemannian metric corresponding to the isotropic conductivity $\gamma = \gamma(x)I$ in Ω is given by

$$g_{ij}(x) = \det(\gamma(x)I)^{1/(2-n)} (\gamma(x)I)^{-1} = \gamma(x)^{2/(n-2)}\delta_{ij}.$$

Then, if Δ_g is the Laplace–Beltrami–operator corresponding to the metric g, we have $\Delta_g = |g|^{-1/2}\nabla \cdot \gamma\nabla$m where $|g| = \det(g_{ij})$. This metric is an invariant object and in the deformation F_m it is transformed to the metric $\widetilde{g} = (F_m)_*g$ in $\widetilde{\Omega}$. By [22], the Dirichlet-to-Neumann map $\Lambda_{\widetilde{\gamma}}$ determines the metric tensor $\widetilde{g}_{ij}$ on the boundary in the boundary normal coordinates. In particular, if we consider $\partial\widetilde{\Omega}$ as a submanifold of $\mathbb{R}^n$ with the metric $\widetilde{h} = \widetilde{i}^*(\widetilde{g})$ inherited from $(\widetilde{\Omega},\widetilde{g})$ where $\widetilde{i} : \partial\Omega \to \Omega$ is the identity map, we see that our boundary data determines the metric $\widetilde{h}$ on $\partial\widetilde{\Omega}$. Let now metric $h = i^*(g)$ be the corresponding metric on $\partial\Omega$, where $i : \partial\Omega \to \Omega$ is the identical embedding. Then we have

(4.15) $$\widetilde{h} = (f_m)_*h, \quad h = \gamma^{2/(n-2)}h^E$$

where h^E is the Euclidean metric of $\partial\Omega$. Denote by $\widetilde{h}^E = (f_m)_*h^E$ the metric tensor on $\partial\widetilde{\Omega}$, i.e. the push–forward of the Euclidean metric of $\partial\Omega$ by f_m. Recall that dS_E and $d\widetilde{S}_E$ are the Euclidean volume forms of $\partial\Omega$ and $\partial\widetilde{\Omega}$, respectively. Then the Riemannian volume forms $dS_{\widetilde{h}}$ and dS_h of the metrics $\widetilde{h}$ and h respectively satisfy

$$dS_{\widetilde{h}} = (f_m)_*(dS_h) = \gamma(f_m^{-1}(\widetilde{x}))(f_m)_*(dS_E) = (\gamma\beta) \circ f_m^{-1}(\widetilde{x}) \, d\widetilde{S}_E$$

on $\partial\widetilde{\Omega}$. As β was already determined, this shows that we can find $\gamma(f_m^{-1}(\widetilde{x}))$, $\widetilde{x} \in \partial\widetilde{\Omega}$ and hence by (4.15) we can determine the metric

$$\widetilde{h}^E = \gamma(f_m^{-1}(\widetilde{x}))^{-2/(n-2)}\widetilde{h}.$$

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In other words, if we consider $\partial\widetilde{\Omega}$ as an abstract manifold that can be embedded to $\partial\Omega \subset \mathbb{R}^n$, we have found the metric tensor on $\partial\widetilde{\Omega}$ corresponding to the Euclidean metric of $\partial\Omega$. By the Cohn-Vossen rigidity theorem, intrinsically isometric C^2-smooth surfaces that are boundaries of a strictly convex body are congruent in a rigid motion. For uniqueness, see e.g. [30, Thm. V and VI] and also [11, 12]. This means that the boundary data determines uniquely the map $T \circ f_m^{-1}$, where T is a rigid motion. Hence we can find the surface $T(\partial\Omega)$ and on it the map $T_*\Lambda_{\widetilde\gamma} = T_*\Lambda_\gamma$. Using the uniqueness of of the isotropic inverse problem [35, 24], we see that the boundary data determines $\gamma \circ T^{-1}$. $\qquad\square$

Note that the construction of the surface $\partial\Omega \subset \mathbb{R}^3$ from the intrinsic metric h^E is a more delicate issue, see [26, 29], hence we take care to avoid it.

5. A reconstruction algorithm and the use of conformal flatness.

In this section we consider the case $n = 3$, even though the considerations could be generalized for $n \geq 4$ by changing the Cotton-York tensor to Weyl tensor in our considerations (see Appendix). As noted before, an actual construction of the isometric embedding of an abstract manifold to Euclidean space is complicated and thus we try to avoid it.

We want to find an anisotropic conductivity η such that $R_{\widetilde z, \eta} = \widetilde R$ assuming that $\widetilde R = (f_m)_* R_{z,\gamma}$ where γ is an isotropic conductivity. Clearly, when $F_m : \Omega \to \widetilde\Omega$ is diffeomorphism satisfying $F_m|_{\partial\Omega} = f_m$, the anisotropic conductivity $(F_m)_*\gamma$ is a solution of the inverse problem, but it is not unique. However, we also know that $(F_m)_*\gamma$ has a conformally flat structure and this fact will help in solving the inverse problem as we will see. Note that in principle, one could start to solve the inverse problem by minimizing over all pairs (Ω, σ) of smooth domains $\Omega \subset \mathbb{R}^n$ and all isotropic conductivities σ in Ω. However, the minimization over domains is complicated, and our objective is to find a reasonably simple minimization algorithm where we minimize over conductivities in the fixed model domain $\widetilde\Omega$ with an appropriately chosen cost function.

Let $\eta = (F_m)_*\gamma$ be a possibly anisotropic conductivity in $\widetilde\Omega$ such that γ is isotropic. As already noted it defines a Riemannian metric g on $\widetilde\Omega$, given by

$$[g_{jk}]_{j,k=1}^n = ([g^{jk}]_{j,k=1}^n)^{-1}, \quad g^{jk} = \det(\eta)^{1/(n-2)}\eta^{jk}$$

From now on we will use the Einstein summation convention and omit the summation symbols. As $F_m^{-1} : \widetilde\Omega \to \Omega$ can be considered as coordinates, we see that in proper coordinates the metric g is a scalar function times Euclidean metric, that is, g is conformally flat. This means that

$$g_{ij}(x) = e^{-2\sigma(x)}\overline g_{ij}(x)$$

where $\overline g_{ij}(x)$ is a metric with zero curvature tensor (i.e. flat metric) and $\sigma(x) \in \mathbb{R}$. By [9] (for original work, see [8]), the conformal flatness of the metric g in three dimensions is equivalent to the vanishing of the Cotton-York tensor $C = C_{ij}$ corresponding to g (see Appendix). Note that we can choose $\sigma = \frac{1}{2-n}\log\gamma$ and $\overline g = (F_m)_*(\delta_{ij})$. By [9, formulae (28.18) and (14.1)], σ satisfies a differential equation (with $n = 3$)

$$(5.16)\quad \sigma_{ij} = -\frac{1}{n-2}\mathrm{Ric}_{ij} + \frac{1}{2(n-1)(n-2)}g_{ij}\mathrm{R} - \frac{1}{2}g_{ij}g^{lm}\sigma_l\sigma_k, \quad i,j = 1,\ldots,n$$

where Ric_{ij} is the Ricci curvature tensor of g, R is the scalar curvature or g, and

$$\sigma_k = \frac{\partial \sigma}{\partial x^k}, \quad \sigma_{ij} = \nabla_{e_i}\sigma_j - \sigma_i\sigma_j, \quad \text{where} \quad e_i = \frac{\partial}{\partial x^i},$$

where ∇_{e_i} is the covariant derivative with respect to metric g. Thus if g is given, (5.16) is a second order nonlinear differential equation for σ. By [9, p. 92], the equations (5.16) satisfy the sufficient integrability conditions to be locally solvable if and only if the Cotton-York tensor vanishes. Note that the existence of the isotropic conductivity γ in Ω gives a solution for these equations.

Consider now the following algorithm:

Data: Assume that we are given $\partial\Omega_m$, $\widetilde{R} = (f_m)_* R_{\gamma,z}$ and $z \circ f_m^{-1}$ on $\partial\Omega_m$.

Aim: We look for a metric $\widetilde{g}$ corresponding to the conductivity $\widetilde{\gamma}$ and $\widetilde{z}$ such that on $\partial\Omega_m$ $\widetilde{R} = R_{\widetilde{\gamma},\widetilde{z}}$ and $\widetilde{z} = (f_m)_* z$.

Algorithm:

1. Determine the two leading terms in the symbolic expansion of $\widetilde{R}$. They determine a contact impedance $\widehat{z}$ and a metric $\widehat{g}$ on $\partial\widetilde{\Omega}$ such that if $\widetilde{R} = R_{\widetilde{\gamma},\widetilde{z}}$ then $\widetilde{z} = \widehat{z}$ and $\widetilde{i}^*(\widetilde{g}) = \widehat{g}$.

2. Form the ratio of reconstructed i.e. $\widehat{z}$, and measured contact impedances

$$\widehat{r}(\widetilde{x}) := \frac{z(f_m^{-1}(\widetilde{x}))}{\widehat{z}(\widetilde{x})}, \quad \widetilde{x} \in \partial\widetilde{\Omega}.$$

Note that then

$$\widehat{r}(\widetilde{x})(f_m)_*(dS_E) = d\widetilde{S}_E$$

since the contact impedances transformed as densities.

3. Let $dS_{\widehat{g}}$ be the volume form of $\widehat{g}$ on $\partial\widetilde{\Omega}$. Then

$$dS_{\widehat{g}} = (\det \widehat{g})^{1/2}\, d\widetilde{S}_E.$$

Define

$$\widehat{\gamma} = (\det \widehat{g})^{1/2}\, \widehat{r}.$$

With this choice $\widehat{\gamma}$ will satisfy $\widehat{\gamma}(\widetilde{x}) = \gamma(f_m^{-1}(\widetilde{x}))$ for $\widetilde{x} \in \partial\widetilde{\Omega}$.

4. Define the boundary value $\widehat{\sigma}$ for the function σ by

$$\widehat{\sigma}(\widetilde{x}) = \frac{1}{2-n}\log\left(\widehat{\gamma}(\widetilde{x})\right), \quad \widetilde{x} \in \partial\widetilde{\Omega}.$$

5. Solve the minimization problem

$$\min F_\tau(\widetilde{z}, \widetilde{\sigma}, \widetilde{\gamma}) + \alpha\, H(\widetilde{z}, \widetilde{\sigma}, \widetilde{\gamma})$$

where $H(\widetilde{z}, \widetilde{\gamma})$ is a regularization functional, say

$$H(\widetilde{z}, \widetilde{\sigma}, \widetilde{\gamma}) = \|\widetilde{z}\|_{H^8(\widetilde{\Omega})} + \|\widetilde{\gamma}\|^2_{H^8(\widetilde{\Omega})} + \|\widetilde{\sigma}\|^2_{H^8(\widetilde{\Omega})},$$

$\alpha \geq 0$ is a regularization parameter, and

$$
\begin{aligned}
F_\tau(\widetilde{z}, \widetilde{\sigma}, \widetilde{\gamma}) \;=\; & \|\widetilde{R} - R_{\widetilde{\gamma}, \widetilde{z}}\|^2_{L(H^{-1/2}(\partial\widetilde{\Omega}))} \\
& + \left\|\frac{\widetilde{z}(\widetilde{x})}{z(f_m^{-1}(\widetilde{x}))} - \widehat{r}(\widetilde{x})\right\|_{L^2(\partial\widetilde{\Omega})} \\
& + \tau \|C\|^2_{L^2(\widetilde{\Omega})} \\
& + \|\widetilde{\sigma}|_{\partial\widetilde{\Omega}} - \widehat{\sigma}\|^2_{L^2(\partial\widetilde{\Omega})} \\
& + \sum_{i,j=1}^n \left\|\widetilde{\sigma}_{ij} - \left(-\frac{1}{n-2}\mathrm{Ric}_{ij} + \frac{1}{2(n-1)(n-2)}\widetilde{g}_{ij}R - \frac{1}{2}\widetilde{g}_{ij}\widetilde{g}^{lm}\sigma_l\sigma_k\right)\right\|^2_{L^2(\widetilde{\Omega})}
\end{aligned}
$$

where $\tau \geq 0$, $\widetilde{g}$ is the metric tensor corresponding to $\widetilde{\gamma}$, $C = C_{ij}$ is the Cotton-York tensor of $\widetilde{g}$, and finally Ric and R are the Ricci curvature and scalar curvature tensors respectively of $\widetilde{g}$.

Note that above value of the Cotton-York tensor at $x \in \Omega$, $C_{ij}(x)$, the Ricci curvature tensors $R_{ij}(x)$, and the scalar curvature $R(x)$ depend on the values of the conductivity η and its three first derivatives at x.

PROPOSITION 5.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, strictly convex, C^∞–domain. Assume that $\gamma \in C^\infty(\overline{\Omega})$ is an isotropic conductivity, $z \in C^\infty(\partial\Omega)$, $z > 0$ be a contact impedance and $R_{\gamma, z}$ be the corresponding Robin-to-Neumann map. Let $\widetilde{\Omega}$ be a model of the domain satisfying the same regularity assumptions as Ω, and $f_m : \partial\Omega \to \partial\widetilde{\Omega}$ be a C^∞–smooth diffeomorphism.*

Assume that we are given $\partial\widetilde{\Omega}$, the values of the contact impedance $z(f_m^{-1}(\widetilde{x}))$, $\widetilde{x} \in \partial\widetilde{\Omega}$, and the map $\widetilde{R} = (f_m)_* R_{\gamma, z}$.

Let $\tau \geq 0$. Then minima of $F_\tau(\widetilde{z}, \widetilde{\sigma}, \widetilde{\gamma})$ is zero and any minimizers $\widetilde{z}$, $\widetilde{\sigma}$ and $\widetilde{\gamma}$ of $F_\tau(\widetilde{z}, \widetilde{\sigma}, \widetilde{\gamma})$ satisfy $\widetilde{z} = (f_m)_* z$ and there is a diffeomorphism $\widetilde{F} : \Omega \to \widetilde{\Omega}$ such that $\widetilde{F}|_{\partial\Omega} = f_m$, $\widetilde{\gamma} = \widetilde{F}_* \gamma$ and $\widetilde{\sigma} = -\log \widetilde{\gamma}$.

Proof. Assume first that $\tau > 0$. The minimizer exists because of existence of Ω, γ, z and σ, and the minimum is zero. Let $\widetilde{z}$, $\widetilde{\sigma}$ and $\widetilde{g}$ be some minimizers of F_τ. As then the Cotton-York tensor is zero and the equations (5.16) are valid, if follows from [9], that the metric $\overline{g}_{ij} = \exp(2\sigma(\widetilde{x}))g_{ij}(\widetilde{x})$, $x \in \widetilde{\Omega}$ is flat. Since $R_{\widetilde{z}, \widetilde{\gamma}} = \widetilde{R}$, we have $\widetilde{z} = (f_m)_* z$ and the metric $\widetilde{g}$ corresponding to $\widetilde{\gamma}$ has to satisfy on the boundary $i^*\widetilde{g} = \widehat{g}$. This, and the vanishing of F_τ imply that

$$
\begin{aligned}
i^*\overline{g} \;=\; & \exp(2\widehat{\sigma})i^*\widetilde{g} = \exp(2\widehat{\sigma})\widehat{g} = \exp(2\widehat{\sigma})(f_m)_*(\gamma h_E) = \\
\;=\; & \exp(2\widehat{\sigma})\,\widehat{\gamma}\,(f_m)_*(h_E) = (f_m)_*(h_E).
\end{aligned}
$$

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Consider now $(\widetilde{\Omega}, \overline{g})$ as a Riemannian manifold. As $\overline{g}$ is flat, we know that $(\widetilde{\Omega}, \overline{g})$ can be embedded isometrically to domain $\Omega_0 \subset \mathbb{R}^n$. Let $k : \widetilde{\Omega} \to \Omega_0$ be this embedding. Since $i^*\overline{g} = (f_m)_*(h_E)$, it follows from the Cohn-Vossen rigidity theorem that the boundary $\partial\Omega_0$ and $\partial\Omega$ are congruent in a rigid motion T and $k \circ f_m = T|_{\partial\Omega}$. Then $(T^{-1} \circ k)_*\widetilde{\gamma}$ is isotropic conductivity, the contact impedances of $(T^{-1} \circ k)_*\widetilde{z}$ and z coincide, and the Robin-to-Neumann maps of $(T^{-1} \circ k)_*\widetilde{\sigma}$ and σ coincide. By the uniqueness of the isotropic inverse conductivity problem [35], $(T^{-1} \circ k)_*\widetilde{\gamma} = \gamma$. This proves the claim in the case $\tau > 0$.

Next, consider the case $\tau = 0$. Again, minimizer exists because of existence of Ω, γ, z and σ, and the minimum is zero. Let $\widetilde{z}$, $\widetilde{\sigma}$ and $\widetilde{g}$ be some minimizers. As the minima of F_τ is zero, the equations (5.16) are valid. By [9, p. 92], the solutions σ satisfy the integrability conditions

$$(5.17) \qquad \nabla_k \sigma_{ij} - \nabla_j \sigma_{ik} = \sigma_l \mathrm{R}^l_{ijk}, \quad i, j, k = 1, \ldots, n$$

that imply that the conformal covariant satisfies R_{ijk} vanishes. Thus the Cotton-York tensor C_{ij} is zero. This means that the minimizers $\widetilde{z}$, $\widetilde{\sigma}$ and $\widetilde{g}$ of F_τ with $\tau = 0$ are also minimizers of F_τ with any $\tau > 0$. $\qquad\square$

One can think of τ as a regularization parameter: in general the solvability properties of equations (5.16) are sensitive to the compatibility conditions, i.e. the vanishing of the Cotton–York (or the Weyl tensor in higher dimensions).

To find the domain Ω, we can continue the above algorithm by applying the fact that conformally Euclidean manifold of dimeision n can be conformally embedded to $\mathbb{R}^n$ in a constructive way (cf. [19]).

6. In the previous steps 1.–5. we have found metric tensors $\widetilde{g}$ and $\overline{g} = e^{2\widetilde{\sigma}}\widetilde{g}$ on $\widetilde{\Omega}$ such that $\widetilde{g} = F_*(g)$ and $\overline{g} = F_*(g^E)$ where g is the metric corresponding to the metric γ on Ω, g^E is the Euclidean metric on Ω, and $F : \Omega \to \widetilde{\Omega}$ is some diffeomorphism.

Let $y \in \widetilde{\Omega}$ and find geodesics $\overline{\mu}_{y,\xi}(s)$ starting from y with respect to the metric $\overline{g}$. We parametrize these geodesics in such a way that $\overline{\mu}_{y,\xi}(0) = y$ and $\partial_s\overline{\mu}_{y,\xi}(0) = \xi$ is a unit tangent vector of the tangent space $(T_y\widetilde{\Omega}, \overline{g})$. These geodesics correspond to the halflines in $\mathbb{R}^3$ starting from some point $y_0 \in \Omega$. Let $J : (T_y\widetilde{\Omega}, \overline{g}) \to \mathbb{R}^3$ be a linear isometry and define a map $\kappa : \widetilde{\Omega} \to \mathbb{R}^3$ by setting

$$\kappa(\overline{\mu}_{y,\xi}(s)) = s\, J\xi, \quad s \geq 0.$$

Then $\kappa \circ F : \Omega \to \mathbb{R}^3$ is an affine isometry that extends to a rigid motion $T : \mathbb{R}^3 \to \mathbb{R}^3$ with $T(y) = 0$. Thus we can find $\kappa(\widetilde{\Omega}) = T(\Omega)$, $\kappa_*(\widetilde{\sigma}) = T_*\gamma$, and $\kappa_*(\widetilde{z}) = T_*z$

Thus we have shown the followig reconstruction result.

COROLLARY 5.2. *Let $\Omega, \gamma, z, \widetilde{\Omega}$ and f_m be as in Proposition 5.1. Assume that we are given $\partial\widetilde{\Omega}$, the contact impedance $z(f_m^{-1}(\widetilde{x}))$, $\widetilde{x} \in \partial\widetilde{\Omega}$, and the Robin-to-Neumann map $\widetilde{R} = (f_m)_*R_{\gamma,z}$. Then the algorithm 1.–6. determines Ω, γ, and z up to a rigid motion $T : \mathbb{R}^3 \to \mathbb{R}^3$.*

We intend to investigate the numerical implementation of the method and give numerical test results in part II of this paper.

Appendix. Here we define the conformal curvature tensors. We say that a metric g_{ij} in a domain $\Omega \subset \mathbb{R}^n$ is conformally flat if there is a scalar function $a(x) > 0$ such that the curvature of tensor of $a(x)g_{ij}(x)$ is identically zero.

First, let γ be an isotropic conductivity, i.e., a smooth positive function in Ω and $F : \Omega \to \widetilde{\Omega}$ be a diffeomorphism. Let $\eta = F_*\gamma$ be a possibly anisotropic conductivity in $\widetilde{\Omega}$. It defines a Riemannian metric $\widetilde{g}$ on $\widetilde{\Omega}$, given by

$$[\widetilde{g}_{jk}]_{j,k=1}^n = ([\widetilde{g}^{jk}]_{j,k=1}^n)^{-1}, \quad \widetilde{g}^{jk} = \det(\eta)^{1/(n-2)}\eta^{jk}$$

As $F^{-1} : \widetilde{\Omega} \to \Omega$ can be considered as coordinates, we see that in proper coordinates the metric $\widetilde{g}$ is a scalar function times Euclidean metric, that is, $\widetilde{g}$ is conformally flat.

Next we consider a general metric tensor g_{ij} and recall facts concerning its conformal flatness Note that below we use the Einstein summation convention and omit the summation symbols when possible. The following tensors are related to conformal flatness.

(a) Assume that $n = 3$. Then the conformal covariant, given in terms of curvature tensors (see the explanation on notations below) is

$$R_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{2(n-1)}(g_{ik}\nabla_j R - g_{ij}\nabla_k R).$$

In the three dimensional case R_{ijk} defines a tensor that can be considered as vector valued 2-form $R_{ijk}dx^j \wedge dx^k$. Operating with the Hodge operator $*$ to this 2-form, we obtain the Cotton-York tensor,

$$\mathrm{C}_{ij} = g^{kp}g^{lq}\nabla_k(R_{li} - \frac{1}{4}R\,g_{li})\epsilon_{pqj},$$

where ϵ_{pqj} is the Levi-Civita permutation symbol.

(b) Assume that $n \geq 4$. Then the Weyl tensor is

$$
\begin{aligned}
W_{ijkl} &= R_{ijkl} + \frac{1}{n-2}(g_{il}R_{kj} + g_{jk}R_{kj} - g_{ik}R_{lj} - g_{jl}R_{ki}) + \\
&\quad + \frac{1}{(n-1)(n-2)}(g_{ik}g_{lj} - g_{il}g_{kj})R.
\end{aligned}
$$

The crucial fact related to our considerations is that the metric g is conformally flat if and only if in the dimension $n = 3$ the Cotton-York tensor vanishes and in the dimension $n = 4$ the Weyl tensor vanishes, see [9, p. 92] or [8, 36, 2].

Above, R_{ijkl} is the Riemannian curvature tensor,

$$R_{ijkl} = \frac{\partial}{\partial x^k}\Gamma^i_{jl} - \frac{\partial}{\partial x^l}\Gamma^i_{jk} + \Gamma^p_{jl}\Gamma^i_{pk} - \Gamma^p_{jk}\Gamma^i_{pl}, \quad R^p_{jkl} = g^{pi}R_{ijkl}$$

where Γ^i_{jk} are Christoffel symbols,

$$\Gamma^i_{jk} = \frac{1}{2}g^{pi}(\frac{\partial g_{jp}}{\partial x^k} + \frac{\partial g_{kp}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^p}),$$

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R_{ij} is the Ricci curvature tensor, $R_{ij} = R_{ijk}^k$, and R is the scalar curvature $R = g^{ij}R_{ij}$. Finally, above ∇_k is the covariant derivative that is defined for a (0,2)-tensor A_{il} and a (0,1)-tensor B_l by

$$\nabla_k A_{li} = \frac{\partial}{\partial x^k} A_{li} - \Gamma_{kl}^p A_{pi} - \Gamma_{ki}^p A_{lp}, \qquad \nabla_k B_l = \frac{\partial}{\partial x^k} B_l - \Gamma_{kl}^p B_p.$$

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